Math 255A' Lecture 20 Notes

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1 Positive Operators and Spectral Families

1.1 Positive operators

We want to generalize the following theorem, without the assumption of compactness.

Theorem 1.1 (Spectral theorem in finite dimensions)**.** *Let* $\dim(H) < \infty$*, and let* $T : H \to H$ *be a self-adjoint operator with eigenvalues* $a \leq \lambda_1 < \lambda_2 < \cdots < \lambda_m = b$*. Then*

$$T = \sum_{i=1}^{n} \lambda_i P_{\lambda_i},$$

where P_{λ_i} *is the projection onto* $\ker(T - \lambda_i)$*.*

Example 1.1. On $L^2([0,1])$ we have $Tf(x) = xf(x)$, the multiplication operator. Then $\|T\|_{\mathrm{op}} \leq 1$, and

$$\langle Tf, g \rangle = \int_0^1 \overline{x} f(x) \overline{g(x)} \, dx = \langle f, Tg \rangle .$$

However, T has no eigenvectors! If $Tf = \lambda f$, then $xf(x) = \lambda f(x)$ for a.e. x. So $f = 0$ a.e.

Observe that if $V = \ker(T - \lambda) \neq \{0\}$, then V is reducing and $T|_V = \lambda I_V$. We want to loosen this to $\mu I_V \leq T|_V \leq \lambda I_V$ for $\mu < \lambda$.

Definition 1.1. $T \in \mathcal{B}(H)$ is **positive** (written $T \geq 0$) if T is self-adjoint and $\langle Tx, x \rangle \geq 0$. If S, T are self-adjoint, we say $S \leq T$ if $T - S \geq 0$.

This defines a partial order on the set of self-adjoint operators. How does this relate to our previous examples?

Example 1.2. In the finite dimensional case, for $\lambda \in \mathbb{R}$, define

$$E(\lambda) := \sum_{i : \lambda_i \leq \lambda} P_{\lambda_i}, \qquad E(\mu, \lambda) := \sum_{\mu < \lambda_i \leq \lambda} P_{\lambda_i} = E(\lambda) - E(\mu).$$

These all reduce T, and

$$\mu E(\mu,\lambda) \leq TE(\mu,\lambda) \leq \lambda E(\mu,\lambda)$$

for all $\mu \leq \lambda$. If λ_i is the unique element of $\sigma_p(T) \cap (\mu,\lambda]$, $\lambda_i P_i \leq TP_{\lambda_i} \leq \lambda_i P_i$.

Example 1.3. With the multiplication operator T on L^2, let $V(\mu,\lambda) := \{f \in L^2([0,1]) : f = f\mathbb{1}_{(\mu,\lambda]}\}$ for any $\mu \leq \lambda$. Then let $E(\mu,\lambda) = P_{V(\mu,\lambda)}$. We can check that

$$TE(\mu,\lambda)f(x) = xf\mathbb{1}_{(\mu,\lambda]}(x).$$

Then $\mu E(\mu,\lambda) \leq TE(\mu,\lambda) \leq \lambda E(\mu,\lambda)$.

Lemma 1.1. *Let T be self-adjoint, and let $a = \inf_{\|x\|=1} \langle Tx, x\rangle$. and $b = \sup_{\|x\|=1} \langle Tx, x\rangle$. Then $a \leq T \leq b$ and $\|T\| = \max(|a|,|b|)$.*

Proof. If $\|x\| = 1$, then

$$\langle (T-a)x, x\rangle = \langle Tx, x\rangle = a \geq 0.$$

The upper bound is the same.

We have seen already that $\|T\| = \sup |\langle Tx, x\rangle|$. □

Corollary 1.1. *If $S \leq T$ and $T \leq S$ then $S = T$.*

Proof. This implies that $\langle (S-T)x, x\rangle = 0$ for all x. So the norm is $\|S-T\| = 0$. □

Lemma 1.2. *For projections P, Q, the following are equivalent:*

1. $P \leq Q$.
2. $QP = PQ = P$.
3. $Q - P$ *is a projection.*
4. $\|Px\| \leq \|Qx\|$.
5. $\operatorname{ran} P \subseteq \operatorname{ran} Q$.

Proof. (1) $\implies$ (5): If (5) is false, then there is some $x \neq 0$ such that $Px = x$ but $Qx \neq x$. Then $\|x\|^2 = \langle Px, x\rangle$, but $\langle Qx, x\rangle = \|Qx\|^2 < \|x\|^2$. This contradicts (1).

(5) $\implies$ (2): $QP = P$ by the condition of (5), and we get $(QP)^* = P^*Q^* = PQ$ by self-adjointness.

(2) $\implies$ (4): $\|Px\| = \|PQx\| \leq \|Qx\|$.

(2) $\implies$ (3): $\langle (Q-P)x, x\rangle = \langle Q(1-P)x, x\rangle = \langle Q(1-P)x, Qx\rangle \geq 0$.

(3) $\implies$ (1): $Q - P$ is a projection, so $Q - P \geq 0$. □

1.2 Spectral families and the spectral theorem

Definition 1.2. A **spectral family** on H is a map $\lambda \mapsto E(\lambda)$ from $\mathbb{R} \to \{\text{proj. on } H\}$ such that

1. If $\lambda > \mu$, then $E(\lambda) \geq E(\mu)$
2. There exist $a, b \in \mathbb{R}$ such that $E(\lambda) = 0$ if $|lambda < a$ and $E(\lambda) = I$ if $\lambda \geq b$.
3. $E(\lambda)x \to E(\mu)x$ as $\lambda \downarrow \mu$ for all $x \in H$ (convergence in the strong operator topology).

Theorem 1.2. *Let T be a self-adjoint operator on H. Then there exists a spectral family $(E(\lambda))_{\lambda \in \mathbb{R}}$ such that*

$$a = \inf_{\|x\|=1} \langle Tx, x \rangle, \qquad b = \sup_{\|x\|=1} \langle Tx, x \rangle$$

$$T = \int_{\mathbb{R}} \lambda \, dE(\lambda).$$

This means $\langle Tx, y \rangle = \int_{[a,b]} \lambda \, d\mu_{x,y}$ for all $x, y \in H$, where $\mu_{x,y}$ is the Lebesgue-Stieltjes measure corresponding to $F_{x,y}$.

To interpret this integral, we need the following lemma.

Lemma 1.3. *If E is a spectral family, then for any $x, y \in H$, then function $F_{x,y} : \lambda \mapsto \langle E(\lambda)x, y \rangle$ is right-continuous and of bounded variation.*

Proof. Right continuity follows from property (3) of a spectral family. For bounded variation,

Step 1: If $y = x$, then $F_{x,x}(\lambda) = \|E(\lambda)x\|^2$, which is increasing with λ.

Step 2:

$$F_{x,y}(\lambda) = \langle E(\lambda)x, y \rangle = \frac{\langle E(\lambda)(x+y), x+y \rangle - \langle E(\lambda)x, x \rangle - \langle E(\lambda)y, y \rangle}{2}$$

is a difference of nondecreasing functions, so it is of bounded variation. □

Example 1.4. In the finite dimensional case, $E(\lambda)$ is constant, except for finitely many jumps. So the integral becomes a finite sum.

Example 1.5. Returning to the multiplication operator on L^2, if $f, g \in L^2([0,1])$, then

$$\langle Tf, g \rangle = \int_0^1 x f(x) \overline{g(x)} \, dx, \qquad dx = d\mu_{f,g}.$$

Here, $E(\lambda)$ is the proejction onto $\{f = f \mathbb{1}_{[0,\lambda]}\}$,and $\langle E(\lambda)f, g \rangle = \int_0^\lambda f \overline{g} \, dx$.

1.3 Functional calculus

How do we find this map $\lambda \mapsto E(\lambda)$? In the finite dimensional case, we have a self-adjoint T with eigenvalues $a = \lambda_1 < \lambda_2 < \cdots < \lambda_m = b$ and $T = \sum_i \lambda_i P_{\lambda_i}$. If $p(t) = \sum_{j=1}^k x_j t^j \in \mathbb{R}[t]$ is a polynomial, we can write $p(T) = \sum_{j=1}^k c_j T^j$. Since $T^j = \sum_i \lambda_i^j P_i$, we have $p(T) = \sum_i p(\lambda_i) P_{\lambda_i}$.

Choose any $p_\lambda \in \mathbb{R}[t]$ such that

$$p_\lambda(t) = \begin{cases} 1 & t = \lambda_i \leq \lambda \\ 0 & t = \lambda_i > \lambda. \end{cases}$$

Then

$$p_\lambda(T) = \sum_{\lambda_i \leq \lambda} P_{\lambda_i} = E(\lambda).$$

We need to make this work in infinite dimensions. But $\mathbb{R}[t]$ is not rich enough. We must extend the map $\mathbb{R}[T] \to \mathcal{B}(H)$ taking $p \mapsto p(T)$ to a larger class of functions. After doing so, we get the **functional calculus** of T. In particular, we want to be able to get the function $p(T)$, where $p(t) = \mathbb{1}_{(-\infty,\lambda]}(t)$.